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LETTER TO THE EDITOR

# Squeezed states in a quantum chaotic system

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**Abstract.** Time evolution of squeezed states in the quantum kicked rotator model is analysed. Squeezing influences the shape of quantum revivals obtained in the regime of classically regular motion, but does not facilitate the diffusion in angular momentum in the regime of classically chaotic motion. The speed of unlimited energy growth, which occurs in the case of quantum resonance, depends significantly on the squeezing parameter.

Time-dependent studies of classically chaotic quantum systems were typically performed by applying the pure momentum eigenstates [1, 2] or standard coherent states [3, 4]. Problems of detection of gravitational waves suggested a new class of states, the squeezed states, which fulfil the uncertainty relation with reduced dispersion in one variable [5]. Squeezed states have been intensively studied recently in quantum optics (the current status of research on squeezed states is presented in [6]). In this letter the behaviour of squeezed states in a quantum chaotic system is discussed.

We analyse the time evolution of squeezed states in the frequently studied model of a periodically kicked quantum rotator [1-3, 7]. This system is defined by the Hamiltonian:

$$H = p^2/2 - \frac{K}{2\pi}\cos(2\pi q)\sum_{l=-\infty}^{l=\infty}\delta(t-l) \tag{1}$$

where $q$ and $p$ are scaled angle and angular momentum variables, respectively, $K$ is a kicking strength parameter, the rotator mass and kicking period are equal to 1. Expanding the wavefunction $\Psi$ in terms of the eigenstates of $H_0 = p^2/2$ as

$$\Psi(q, t) = \frac{1}{2\pi}\sum_{n=-\infty}^{n=\infty} a_n(t)\exp(2\pi niq) \tag{2}$$

we obtain a quantum map: $a_n^{(l+1)} = \Sigma_j\, a_j^{(l)} U_n^j$. The infinite matrix of the evolution operator $\tilde{U}$ is given by [1]

$$U_n^j = \exp(-\mathrm{i}j^2 2\pi\tau)\,\mathrm{i}^{n-j} J_{n-j}(k) \tag{3}$$

where $\tau = \hbar/4\pi$, $k = K/4\pi\tau$ and $J_r(x)$ is the ordinary Bessel function of the first kind and order $r$. The kicking strength parameter $K$ governs the motion of the corresponding classical model. For $K \approx 1$ the last KAM orbit breaks down, and the unlimited diffusion in energy takes place (see [8] for a review of the classical standard map). The dynamics of the quantum system is much more complicated, and depends on two relevant parameters. For any rational value of $\tau$ the energy growth caused by the quantum resonance is unbounded in time [9], while for all irrational $\tau$ values, the energy remains limited for any initial conditions and values of $K$ [2].

We construct appropriate squeezed states generalising the Gaussian coherent states introduced by Chang and Shi [3] for the case of periodic variable $q$. With each point

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$\{p_0, q_0\}$ of the classical phase space we associate the corresponding squeezed state $|\Psi^R_{p_0,q_0}\rangle$ defined by the set of expansion coefficients $a_n$

$$a_n := (\pi\, e^{2R})^{-1/4} \exp[i(bc - \sqrt{2}\, bn)] \exp\left[\frac{-1}{e^{2R}}\left(\frac{n}{\sqrt{2}} - c\right)^2\right] \tag{4}$$

where $b = \pi\sqrt{2}q_0$, $c = \pi\sqrt{2}p_0/\hbar$, and $R$ is an arbitrary squeezing parameter. For negative $R$ values the state is squeezed in momentum ($\Delta p < \Delta q$), for positive $R$ values the state is squeezed in the angular coordinate ($\Delta q < \Delta p$), and for $R = 0$ we recover the standard Gaussian coherent states.

Let us consider the case of classically regular motion. For a small value of the perturbation parameter $K$ and the initial point $\{p_0, q_0\}$ lying outside resonances in the phase space the classical orbit forms a slightly distorted line $p = p_0$. In this case the time evolution of the angular momentum exhibits regular oscillations, with their period depending on $p_0$. In order to compare the behaviour of the quantum state localised in the finite volume proportional to $\hbar$, one can take an ensemble of classical points lying in the vicinity of the point $\{p_0, q_0\}$, and analyse the angular momentum averaged over all points of the ensemble.

Since the vibrations of slightly different frequencies dephase, the oscillations of averaged momentum are damped and the average tends to a constant value. In the corresponding quantum system the oscillations of the expectation value of momentum are damped as in the classical model, but after a certain time $T_r$ the oscillations appear again. This phenomenon, called quantum revival [10-12], is caused by the discrete structure of the energy levels. The revival time $T_r$ is proportional to $1/\hbar$, and tends to infinity in the semiclassical limit.

The time evolution of the average momentum $\langle p(t)\rangle$ for the initial state $|\Psi^R\rangle$ localised on the point $\{p_0, q_0\}$, and for three different values of the squeezing parameter $R$, is presented in figure 1. Numerical results show the influence of squeezing on the shape of revivals. For greater $R$ values, squeezing is stronger in angle, so the dispersion of angular momentum is larger. Frequencies of oscillations dephase faster and revivals are more prominent. A similar effect was recently reported in the modified Jaynes-Cummings model [13].

In the classical system for a strong enough perturbation (i.e. $K > 1$) the unbounded diffusion in angular momentum appears [8]. The classical orbits can cross cantori—the remnants of the last KAM tori. In the corresponding quantum system the average momentum remains bounded for an arbitrary large value of the perturbation parameter $K$. Following an intuitive approach, a coherent state occupying finite volume in the phase space is too 'fat' to cross small gaps in cantori. Since it is easier to throw a javelin through a net than a ball of the same volume, one could speculate that the squeezed states may move in the phase space easier than the corresponding coherent states.

Numerical results, however, do not confirm this hypothesis. Even for a very long time of evolution the averaged angular momentum remains in the vicinity of the initial value $p_0$, and does not depend on the squeezing parameter $R$. Those results are not surprising, since during time evolution the squeezed state rotates in the phase space [14] and spreads over neighbouring regions of the phase space, losing its original shape [15].

The analysis of the expansion of an initial quantum state in the basis of the quasienergy eigenstates (eigenstates of the matrix $U$) provides an additional piece of information. The number, $M$, of eigenstates relevant in such an expansion characterises

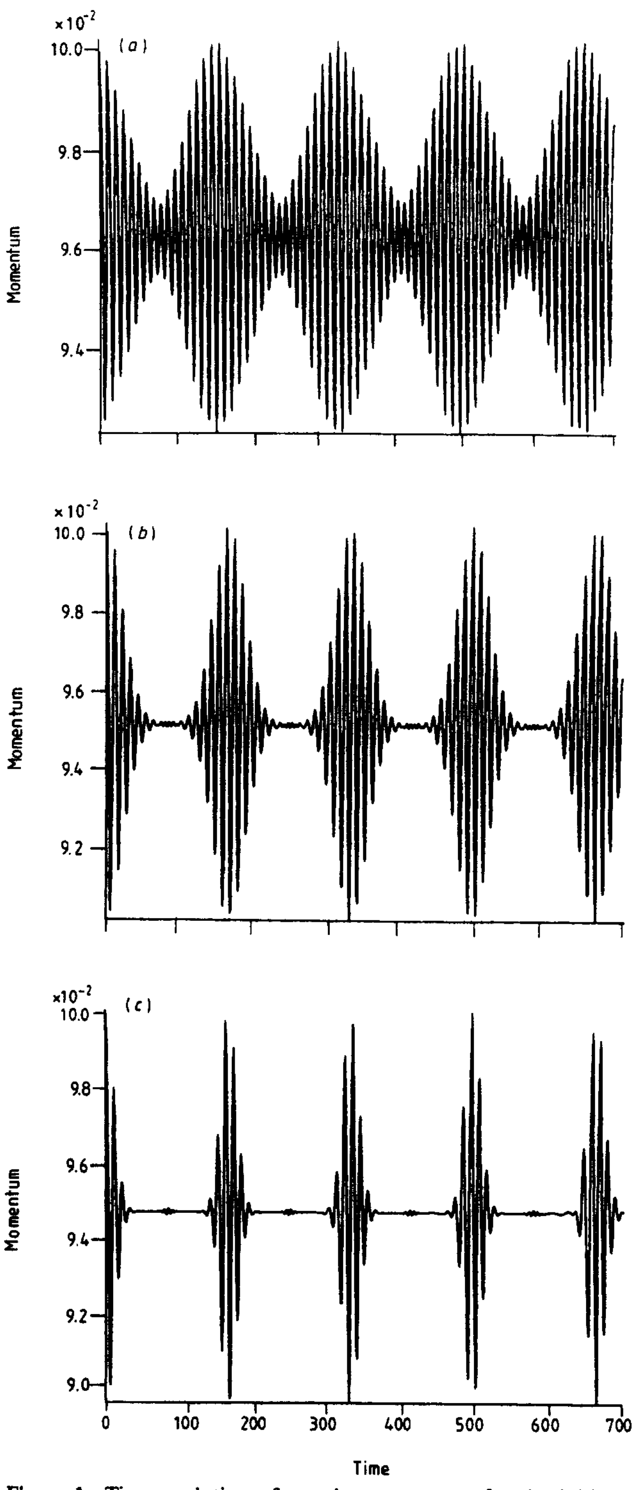


**Figure 1.** Time evolution of angular momentum for the initial state $p_0 = 0.1$, $q_0 = 0.0$; $K = 0.02$, $\hbar = 0.03767$. The squeezing parameter $R$ is equal to ($a$) $-0.22$, ($b$) 0.47 and ($c$) 1.16.

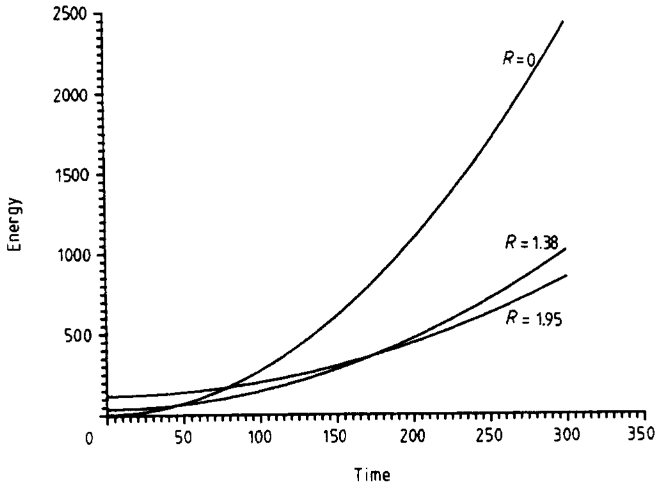


**Figure 2.** Energy growth in the resonance case $\hbar=\pi$ for $K=1.5$, $p_0=0.1$, $q=0.47$ and for three values of the squeezing parameter $R$.

the properties of motion of a quantum state [4]. Our calculations indicate that the number of relevant eigenstates $M$ does not depend significantly on the squeezing parameter $R$, so squeezing cannot change the general properties of the time evolution of a coherent state.

In the particular case of a rational value of the Planck constant $\hbar=r/s$; $r$, $s\in\mathbb{N}$, quantum resonance occurs [9]. The averaged energy of a quantum state grows quadratically in time

$$\langle E\rangle \sim wt^2 \tag{5}$$

and the factor $w$ decreases with an increasing denominator $s$ of the irreducible fraction $r/s$. The energy growth is caused by an increase of the dispersion in momentum $\langle\Delta p\rangle$, while the average momentum $\langle p\rangle$ remains bounded. Thus also in the case of a quantum resonance a coherent state does not move along the momentum axis, but only spreads in the phase space.

As is well known [16], a Gaussian wavepacket representing a single particle spreads during the free evolution, and the dispersion of momentum $\Delta p$ grows with the speed proportional to the initial dispersion of position $\Delta x$. Squeezed states in the system investigated exhibit the similar feature only in the case of quantum resonance: the diffusion factor $w$ decreases with the squeezing parameter $R$—see figure 2. For large dispersion in momentum of the initial state (squeezing in angle) further energy (dispersion) growth is relatively slow. On the other hand, the resonant diffusion occurs faster for initial states squeezed in angular momentum, i.e. for initial states occupying a smaller number of momentum eigenstates $|n\rangle$.

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